Platonism, Structuralism, and Mathematical Applicability

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Abstract

This paper discusses two differing views of mathematics: a modified version of platonism and relativist structuralism. After these positions are detailed, including a discussion on how the traditional reading of Plato makes some ontological mistakes, they are compared for their surprising amount of similarity. The differences end up being primarily metaphysical, with relativist structuralism representing a minimalist version of mathematical platonism. The paper then moves on to a discussion of the two views on mathematical applicability, ultimately reaching the conclusion that the extraordinary explanatory power of mathematics must be justified philosophically. This is a task which has not been satisfactorily completed. This paper also shows that mathematical truths have stronger logical necessity than does the application of mathematical models to physical phenomena. At the end of the paper, we are left with open questions regarding the awe inspiring conformity of physical objects to mathematical objects.

1 Platonism in Mathematics

The evidence in Plato's *Republic* indicates that metaphysics, mathematics, and ethics were all deeply related in Plato's worldview. Beginning with his metaphysics, Plato draws a famous allegory likening man's condition to being strapped immobile in a cave with a source of light from behind, creating a shadow of moving objects on a wall. The strapped prisoners, having been constrained as such for life, accept the shadows on the wall as truly existent objects; to these shadows on the wall Plato likens the realm of human sense perception. In this manner, Plato makes an important distinction between things which are continually coming into being and those that are eternal, which simply are. The former are sensible objects, continually gaining and losing qualities: the same water is hot at some time, cold at some other and is once a gas another time a liquid; because this water is constantly becoming something else, it never truly is. For Plato, the truly immutable objects are forms, general ideas in which all particular objects participate. So there is a square in itself, the properties of which all sensible squares can be said to inherit, albeit in a less than perfect manner.

Mathematics, while useful for practical computations by businessmen, merchants and the like, has much greater ethical importance for Plato because of its ability to help prisoners escape the cave and begin to see objects as they truly are in the light of the Sun. Thus at 518c he writes that, "the instrument with which each learns. . . must be turned around from that which is coming into being together with the whole soul until it is able to endure looking at that which is and the brightest part of that which is." In looking at objects as they truly are, a man will be stronger because he has come closer to understanding the idea of the good which illuminates all the world and thus acts as the source of all knowledge. Plato implores his guardians to "go to calculation and to take it up, not after the fashion of private men, but to stay with it until they [the future guardians] come to the contemplation of the nature of numbers with intellection itself, not practicing it for the sake of buying and selling like merchants or tradesmen, but for war and for ease of turning the soul itself around from becoming to truth and being" [15, 525c]. The above passage implies that the mathematical objects are somehow more real than sensible objects or at the very least that the practice of mathematics helps one understand immutable essences and truths.

To help explain this immutability, Plato makes important ontological and metaphysical claims in his divided line passage at the end of Book VI, a passage in which he makes the principle distinction between the worlds of the visible and of the intelligible. While this paper is most concerned with the relationship between the intelligible and the sensible, the ontology of mathematical objects is most clearly outlined when Plato divides each of the two segments of the line by the same initial ratio. The intelligible realm contains on the one hand a segment which takes seemingly self-evident hypotheses and deduces from such hypotheses certain contingent truths and on the other hand, "in the higher of the two, the soul passes out of hypotheses, and goes up to a principle which is above hypotheses," a principle which is absolutely true [15, 510b]. The former deductive realm is the world of mathematics and the latter is what Plato refers to as dialectic.

The distinction between these two realms of the intelligible has given rise to the modern conception of mathematical platonism, endorsed by Kurt Gödel and many other foundationalists, which views mathematics as a discipline that discovers truths about eternally existing objects. This view directly opposes modern formalism which considers mathematical truths as simply derived theorems in a given formal system as defined by a human being. While I do not here attack the platonistic view that mathematical truths are discovered properties of truly existent objects, I see no reason to accept the widely held view that Plato posited an ontological realm in between sensible objects and forms occupied by mathematical objects.

This view does not have direct textual support in the *Republic*, which provides the most detailed account of Plato's philosophy of mathematics. When discussing the divided line, Plato writes that although students of geometry, arithmetic, etc. "make use of the visible forms and reason about them, they are thinking not of these, but of *the ideals* which they resemble; not of the figures which they draw, but of the absolute square and the absolute diameter, and so on..." [15, 510d, emphasis added]. This quotation yields insight into the nature of mathematical objects and sheds light on how mathematics helps the soul turn from the sensible to the intelligible: mathematics reasons about sensible objects not directly as physics does but in their most ideal, eternally existent form. I use the word 'form' here intentionally, for I believe that by "absolute square" Plato refers to the form of the square. While some Plato scholars would argue that the absolute square floats in an ontological space in between sense perception and intelligible forms, I believe that one must understand this square as a form in that all square sensible objects do approximate the absolute square but are never as perfect.

Thus I contend that Plato conceives of mathematical objects as a subset of all forms and not as occupying an ontological space distinct from that of forms. In fact, I will later show that this subset relationship is only due to the time in which Plato was writing and that some modern conceptions of mathematics allow one to treat all Platonic forms as objects of mathematics.

2 Structuralism in General

By structuralism, I mean the belief that mathematical truths are truths not about precise objects but about relationships among objects. This view also makes an ontological distinction between relationships and precise objects, which Isaacson puts well: "The compelling and immediate reason for rejecting the idea that mathematics is about particular objects is that for any mathematical theory the domain of objects which that theory is taken to be about can always be replaced by a domain consisting of different objects, so long as the second domain has a structure isomorphic to that of the first" [8, p.123]. Consider facts of number theory derived from the natural numbers. One can replace the natural numbers with the even natural numbers of one takes by "+1" in the new system "+2" in the old system (the natural numbers). Any truths derived on the natural numbers will be true over the even numbers with this caveat taken into consideration. While there is no compelling reason to make this specific structural change, the fact that one can do so without any loss of generality means that truths about the natural numbers are not about the numbers themselves since the actual numbers can be exchanged for a subset thereof.

So far, structuralism has merely shifted the focus from particular objects to relational structures among objects, summed up nicely by Isaacons: "The invariance of mathematical truth with respect to isomorphism shows that mathematics is concerned not with objects but with relations between objects, that is to say, with structure" [8, p.123-4]. Accepting this view does little to affect the actual practice of mathematics. While structuralism appears to represent a fundamental change in perspective from platonism and other schools of mathematical thought such as empiricism, it remains remarkably neutral with respect to the ontological status of mathematical objects. Finding implications in this realm is thus left to be pursued by the philosopher of mathematics.

3 Ontology in Structuralism

A mathematician can adopt a structuralist methodology in practice without directly committing to any positive metaphysical claims about mathematical objects since the particular objects can be swapped in and out by isomorphism. Adopting a structuralist methodology leaves open questions such as: What do we mean by reference and truth in mathematics? What can we say about the existence of mathematical objects, including the simplest ones such as the natural numbers? Here I shall examine the implications for structuralism on metpahysics and ontology by comparing and contrasting two schools, formalistic and relativistic, of structuralism.

The most natural response to questions about the philosphical significance of structuralism is to adopt along with it a formalist view of mathematics. Simply put, formalism contends that "What we really deal with in mathematics, or at least in pure mathematics, are just empty signs in the end, i.e., signs used to play certain formal games, but not to be, as such, 'about' anything" [16, p.7]. Thus formalist structuralism adopts the view that mathematics deals only with relations between objects and that these objects are empty signs which have no meaning outside of their structural relationships. This view provides distinctly negative answers to the ontological questions of structuralism and therefore ultimately fails to satisfy. Adopting such a formalism renders the question of the metaphysical nature of mathematical objects almost meaningless by saying that these objects have no positive being. Many mathematicians are intrinsically opposed to formalism (and similarly often attracted to platonism) because of a belief that there is more to mathematics than playing around in a sandbox of human design. The platonistic conception of mathematics as discovery is an attractive view that gets cast aside completely by formalist structuralism. One must at least look for a version of structuralism that yields more positive results in the realms of ontology and metaphysics.

A more fruitful structuralist view is one dubbed *relativist structuralism*; this view holds that for any given area of mathematical study, the choice of model is irrelevant as long as we are consistent with said choice. A model defined as such is specified is a specification of objects passed to an axiomatic system. A simple example to use is Peano Arithmetic (PA), in which one specifies a range S, a distinct element '1' and a successor function s; the rules of arithmetic operations such as '$<$', '$+$', '-', etc. are all then reformulated in terms of s and 1. Recalling the passage from section 2 on the natural versus even numbers, we can rephrase the isomorphism as follows: take as our model M in PA the set $\mathbb{N}$[1], let '1' refer to 1[2] and the successor function $s : \mathbb{N} \to \mathbb{N}$ given by $s(x) = x + 1$. Let M' has as its set $S' = \{2n | n \in \mathbb{N}\}$ and successor function $s : \mathbb{N} \to S'$[3] be given by $s(x) = 2(x + 1)$. Relativist structuralism states that it does not matter whether one chooses M or M' as the model of arithmetic as long as the choice remains consistent. At first glance, this relativism of choice appears to imply a relative notion of truth, namely that truth is defined within each model but not more generally. Because, however, all models of PA are isomorphic to one another, "while truth has been defined in a relative way, a non-relative notion of 'truth in arithmetic' is actually implied: truth in all models of Peano Arithmetic" [16, p.350].

[1] Choosing this set presupposes the existence of the natural numbers. We make this choice for convenience and will discuss further ontological commitments later.

[2] This is an important qualifier because we could let '1' refer to any element of $\mathbb{N}$ and attain the same result.

[3] We can use $\mathbb{N}$ as the domain because $N \sim S'$ by the set-theoretic definition of equivalence where $A \sim B \Leftrightarrow \exists f : A \to B$ so that f is bijective; in fact, our choice of s provides such a function if we define $f(x) = s(x - 1)$, which takes out the "successor" nature of s.

There still appears to be a relative notion of ontology in that '1' does not refer to a specific object until a given model of *PA* is chosen. Thus, when practicing mathematics, whether or not once is conscious of it, there is an implicit assignment of a particular object to the symbols of a model; without considering the choice of model, the meaning of mathematical symbols is ambiguous. In this sense, the relativist structuralist view represents "a case of *systematic referential ambiguity*–in a sense we are always talking about 'my number 1', 'my successor function', etc." [16, p.350, author's emphasis]. This system abiguity manifests itself in modern mathematics, for instance, as the existence of many different abstract algebras. While these different systems are generally used to support a formalist viewpoint, they conform to relativist structuralism in that there is a shared structure underlying all algebras given by a certain set of axioms in the same way that the Peano axioms provide a common basis for all arithmetics.

Underlying the common bases of all these structures is an ontological commitment to the existence of sets,[4] including infinite sets.[5] Additionally, "it is assumed that we can talk about such an infinite set, indeed a variety of such sets, independently from our use of arithmetic language" or else there would be no need to reformulate mathematics in a relativist structuralist manner [16, p.351]. In the *PA* example above, the use of $\mathbb{N}$ also presupposed the existence of particular objects 1, 2, 3, and so on. This measure was adopted for the sake of convenience, for we could use the set of indescript and semi-abstract tick marks $\{|, ||, |||, \ldots\}$ with $s(x) = x|$ to represent the same thing. Seen thusly, the natural numbers are not positively existent entities but depend on one's choice of model for Peano arithmetic.

While the philosophical issue of whether or not the ontological commitment to sets is justified or not is beyond the scope of this paper,[6] I contend that modern mathematical practice, under the relativist structuralist view, is commited to the existence of sets and that the acceptance of Zermelo-Frankel set theory (ZFC) renders the commitments completely natural. Additionally, the adoption of ZFC is almost a non-issue in modern mathematics; except in the farthest reaches of research in the foundations of mathematics, virtually all mathematicians do take ZFC for granted. Relativist structuralism, with ZFC working in the background, renders all mathematical truths reducible to statements about sets because all such truths are relationships of objects, the specific objects being elements of sets. This solution is elegant not only for unification purposes but because facts about sets are entirely axiomatized and so the ontological debate over sets is avoided by assuming their existence.

[4]The commitment to the existence of sets is not necessarily the only such commitment. For example, our discussion appears to also assume the existence of functions that map a set to another one.

[5]For Peano arithmetic, we can get by with assuming only one infinite set, $\mathbb{N}$ for instance. But to make this discussion more generally applicable, relativist structuralism does mandate a commitment to the existence of infinite sets in general.

[6]The Benecarraf and Putnam anthology [10] contains five essays devoted entirely to the concept of set.

4 Adapting Platonism

Furthermore, relativist structuralism with ZFC does not completely renounce platonism since sets are considered actually existent abstract objects. While some "doubt that there is any important difference between the structuralist conception of mathematical objects as 'positions' [elements of sets] and the traditional [platonistic] conception of mathematical objects," the lack of difference runs only as deep as application to mathematical practice and ontology [1, p.10]. That this new view allows mathematical practice to continue virtually unchanged can be viewed as a strength. Additionally, relativist structuralism does not represent a fundamental departure from platonism only in the ontological sense that they both have the view of mathematics as discovering truths about truly existing abstract objects.

Open still, however, is the issue of whether relativist structuralism does depart from platonic metaphysics when one considers the nature of sets as opposed to forms, these being the respective mathematical objects. At 596a, Plato writes that "there is one Form for each set of many things to which we give the same name" which is often cited as the 'one over many' nature of forms. In this sense, sets can function as forms for there is a meaning for terms like 'the set of all integers' or 'the set of all subsets of integers'. The apparent similarity, however, can be misleading. For Plato, particular objects participate in a specific general form whereas the relativist structuralist view denies this particularity because we can choose what particular object to which to assign a name. Regarding integers, Benecerraf writes:

> If numbers are sets, then they must be *particular sets*, for each set is some particular set. But if the number 3 is really one set rather than another, it must be possible to give some cogent reason for thinking so; for the position that this is an unknowable truth is hardly tenable. But there seems to be little to choose among the accounts. Relative to our purposes in giving an account of these matters, one will do as well as another, stylistic preferences aside. [11, p.284-5, emphasis in original]

Recalling the discussion in section 3, we have a different particular object assigned to 3 depending on the choice of M or M'. One might argue: under any model of PA, we can write that $3 = s(s(1))$. After all, this is how one constructs the natural numbers under PA. There seems to be a 'place-holder' in all models of PA which 3 positively represents. This place-holder, however, is not a truly existent object. The search for properties of 3 as such is vacuous: its only property is that it is the second successor of 1. But what is 1? Under any specific model, 1 is a particular object but it has no positive meaning of its own because one can assign any specific element of a particular set to be 1. Thus it appears impossible to determine "what the natural numbers 'really are' or what they should be 'identified with' *in an absolute sense*" [16, p.354, emphasis in original] and so sets do not resemble forms in the sense of the one over many principle.

Similarly, sets are not perfect abstractions of particular objects from which the objects inherit certain properties. An easy way to see this distinction is to consider 'the set of all beds' and 'the form of the bed'. The former, while a positively existing set, is a collection of particular objects whereas the latter is a single abstract object representing the most ideal, truest bed. The set of all beds is simple a collection of particulars whereas the form is a single object with unique properties; the only properties of a set are provided by the axioms of ZFC which are true for any and all sets. Thus, sets gain their uniqueness from differences in the objects collected whereas forms are distinct by nature, each one being a particular abstract object.

With these distinctions between sets and forms taken into consideration, relativist structuralism with ZFC represents something of a 'minimalist platonism': mathematical objects are still truly existent abstract objects but they do not have the same sort of perfection and correspondence to sensible objects[7] as Plato's forms.

5 Implications on Mathematical Applicability

Having considered some of the ontological and metaphysical qualities of mathematical objects, an interesting question becomes: do these objects, and therefore, the truths drawn about them, correspond to physical objects? This is the question of applicability. There exist some contrasting intuitions concerning this question. That mathematics has provided us with so many useful models in physics and has had huge explanatory and predictive power makes one believe that there is applicability. On the other hand, the non-uniqueness of physical models and the fact that our models are perpetually proved incorrect and improved seem to imply a disconnect between the necessity of mathematical truths and physical relationships.

While the correspondence between physical objects and forms in platonism may seem to render mathematics directly applicable in physics, Plato maintains that "if a man, gaping up or squinting down, attempts to learn something of sensible things, I would deny that he ever learns–for there is no knowledge of such things" [15, 529b]. Plato maintains this lack of knowledge based on the metaphysical contingency of physical objects. Forms are the only objects which truly *are* and so knowledge can only be concerned with forms. In this description, mathematics provides one way of achieving knowledge but this knowledge is not of physical objects. Plato does not directly address the issue of applicability, and two views appear to be possible: that because forms are ideal abstractions, the mathematical truths about them do not apply to physical objects which are constantly changing, or that because physical objects participate in forms, some of the properties inherited may be some of the mathematical truths. The former view is more amenable to Plato's metaphysics in general given that one of the principle distinctions between forms and sensible objects is that the former are

[7]There are also forms of non-sensible ideas. For example, there is a form of justice to which, Plato argues, everyone refers when they talk about something being 'just'.

eternal and immutable; eternal truths cannot possibly apply to physical objects which are in constant flux.

Hilary Putnam uses a view similar to the structuralist view to argue for the non-uniqueness of physical models, noticing that "in mathematics the number of ways of expressing what is in some sense the same fact (if the proposition is true) while apparently not talking about the same objects is especially striking" [13, p.297]. An obvious example of the implications of the structural view on physics can be seen in the wave-particle duality of electrons: whether one treats an electron as a particle or as a wave, both of these conceptions have the same explanatory power for the behavior of electrons. Yet, as Putnam points out, "It would be absurd to claim that the sentence 'these is an electron-wave with the wavelength λ' is synonymous with the sentence 'there is a particle electron with the momentum $\frac{\hbar}{\lambda}$ and a totally indeterminate position"' [13, p.298]. Thus in terms of what can be explained about electrons mathematically, it is entirely irrelevant whether one views an electron as a particle or a wave. This example shows that physical models make no claims about the actual existence of the objects but merely the relations of objects. Wigner, even though in awe of the explanatory power of mathematics in physics, still articulates this point well: "As regards... the existence of the earth on which we live and on which Galileo's experiments were performed, the existence of the sun and of all of our surroundings, the laws of nature are entirely silent" [18, p.5].

Scientific structuralism proposes to explain the applicability of mathematics by comparing the application of scientific theories to mathematical structuralism. There exists, however, a very significant disanalogy between the two:

> in physical theorizing we also need the *ontological* distinction between theoretical objects and their physical realization. We need to maintain a level of description in which a physical theory can *talk about* electrons, as theoretical objects, without its having to *be about* electrons, as objects that are physically realized in the world. To talk about electrons (or unicorns) is not thereby to bring them into existence as physical objects [2, p.573, emphasis in original]

This distinction is an important one, requiring that theories apply to physical objects without mandating their physical existence. Ideally, one wants to provide scientific theory with the same level of necessity as mathematical truths. Whether or not this is possible will be addressed shortly.

The problem that needs to be solved is the one of representation: even though the non-uniqueness of structuralism implies that models represent multiple kinds of objects, a model of physical phenomena must also represent particular physical objects. To make the connection between mathematical models and empirical phenomena, an intermediate stage of data models is posited. (Data models are mathematical models that claim to describe the results of physical phenomena.) The step from theoretical models to data models is handled trivially by the shared structure that they have: because both are mathematical models, there should be a level at which both can be reduced to isomorphic relationships of sets. More difficulty, however, arises when trying to maintain that a theory of data represents particular physical objects.

To accomplish this connection, one would hope to find a shared structure between the theory of data and the described physical phenomena. The only positive account for the connection between data models and physical phenomena comes from the structural realists who appeal to a 'no miracles' argument. Their argument runs as follows: if the physical phenomena did not conform to some theory of data, then the correspondence that we find would have to be called a miracle. They reject the possibility of miracles and so "while no detailed account of how the data models come to share structure with the phenomena is given, the possibility (or, indeed, necessity) of making the identification is itself justified by appeal to at least an argument" [2, p.579]. Because no positive account of the shared structure is given, the 'no miracles' argument inadequately 'solves' the applicability problem because its 'solution' does not offer a particular prescription. In fact, that the applicability of mathematics to physics has proved so useful in the past leads Wigner and Hamming to call this effectiveness 'unreasonable'. Wigner concludes his lecture with an optimistic look at the inexplicability of applicability:

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure even though perhaps also to our bafflement, to wide branches of learning. [18, p.14]

That no positive account of how phenomena correspond to theories of data not only makes the term miracle appropriate but also renders physical laws less logically necessary than truths of mathematics. This conclusion represents yet another sense in which platonism and structuralism would agree.[8] Structuralism completely ignores the sort of question being asked; mathematics is not concerned at all with what specific model (i.e. a model of data) can be applied to a truth because multiple models will work by isomorphism. Plato's metaphysics, discussed earlier, renders mathematical truths more logically necessary than any of physics because the objects of math exist necessarily in a way that physical objects do not.

6 Conclusion

Traditional mathematical platonism has been slightly revised and then compared to the relativist structuralist viewpoint. This latter position coordinates

[8] They do not agree completely, however, because Plato considers mathematical truths to be mere 'agreements' and not truths in the absolute sense because mathematics takes for granted certain hypotheses. From 533c: "When the beginning is what one doesn't know, and the end and what comes in between are woven out of what isn't known, what contrivance is there for ever turning such an agreement into knowledge?" Thus for Plato, all mathematical truths are essentially large "if-then" statements.

very well with modern mathematical practice but has one gaping hole: it does not apply to its set-theoretic foundations and therefore offers no justification for the acceptance of set theory. Relativist structuralism turns out to be a minimal version of platonism in that under this view, mathematics still deals with truly existing abstract objects, namely sets. These sets do not have any of the metaphysical perfection that platonic forms do.[9] On the one hand, this lack of perfection can be seen as a liberation from platonic metaphysics while retaining the parts most important to mathematical practice.

Under both of these views, mathematical models of physical phenomena do not have the same force of necessity as mathematical truths because there is no definite way to account for the correlation of phenomena to a theory of data. That physical objects do appear to conform to a specific theory thus becomes something of a miracle. The question of why physical phenomena conform to a theory of data falls under philosophy and not mathematics, given that mathematics makes no positive metaphysical claims about its objects. I personally tend to believe that the corrrespondence between phenomena and data is purely luck. Therefore, for a physical model to have a high degree of necessity, one either must assume that the phenomena correspond directly to a theory of data or use idealized abstractions of physical objects so that the correspondence becomes necessary.

What does all of this mean? That one must remain humble when using mathematics for its physical explanatory power. We can be confident to a certain probability that physical phenomena will continue to conform to a theory of data; never, however, can a physical model have the same necessity of mathematics as long as the final objects represented are particular physical objects. Wigner's sense of awe at the effectiveness of mathematics manifests itself in unanswered philosophical questions: Why does mathematics appear to describe physical phenomena so accurately? Can we ever achieve physical necessity of the same order as logical necessity? Attempts to answer these questions require metaphysical and epistemological commitments that, while fascinating, are the subject of a different paper. For now, we remain in awe of the applicability of mathematics to natural sciences.

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